

# Notes on the computation of periodic orbits using Newton and Melnikov's method: Stroboscopic vs Poincaré map

Albert Granados

## Abstract

These notes were written during the 9th and 10th sessions of the subject Dynamical Systems II coursed at DTU (Denmark) during the Winter Semester 2015-2016. They aim to provide students with a theoretical and numerical background for the computation of periodic orbits using Newton's method. We focus on periodically perturbed quasi-integrable systems (using the forced pendulum as an example) and hence we take advantage of the Melnikov method to get first guesses. However, these well known techniques are general enough to be applied in other type of systems. Periodic orbits are computed by solving a fixed-point equation for the stroboscopic map, which is very fast and precise for hyperbolic periodic orbits. However, for non-hyperbolic ones the method fails and we use the Poincaré map instead. In both cases we show how to compute the Jacobian of the maps, which is necessary for the Newton method, by means of variational equations and the implicit function theorem.

Some exercises are proposed along the notes, whose solutions can be found in [github.com/a-granados](https://github.com/a-granados).

The notes themselves do not contain any reference, although everything described here is well known in the Dynamical Systems community. A typical reference for the Melnikov method for subharmonic orbits is the book [GH83]. More about the variational equations and their numerical applications can be found in the notes [Sim].

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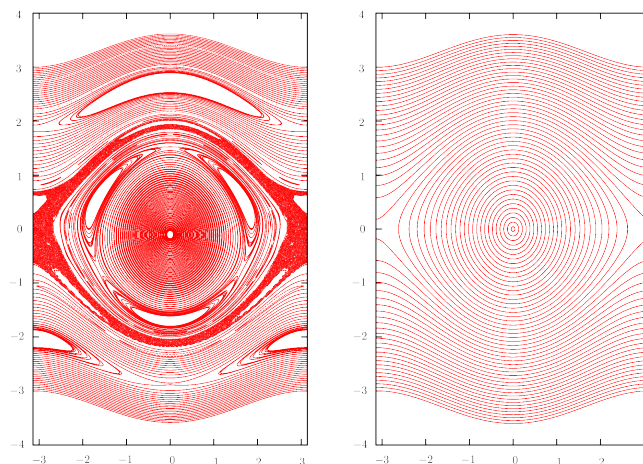


Figure 1: Phase portrait of the unperturbed pendulum (right). Phase portrait for  $\varepsilon > 0$  (left).

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## 1 Introduction

The main goal of these sessions will be to understand the dynamics of a system exhibiting a continuum of periodic orbits when we add a small periodic forcing. The most paradigmatic example is probably the perturbed pendulum; however, such systems massively appear in real applications, specially in celestial mechanics. This has given rise to classical problems exhibiting extremely rich dynamics, such as the restricted three body problem.

In these sessions we will see some theoretical results, but we will mainly visualize them through analytical and numerical computations for a particular example, the forced pendulum:

$$\ddot{u} + \sin(u) = \varepsilon g(x, t), \quad (1)$$

where  $\varepsilon \geq 0$  is a small parameter,  $x = (u, \dot{u})$  and  $g(x, t)$  a periodic forcing:  $g(x, t + T) = g(x, t)$ . We will mainly see that, when  $\varepsilon = 0$  the phase portrait

looks like Figure 1 (right) and that, when  $\varepsilon > 0$  looks like Figure 1 (left). We will learn theoretical and numerical techniques to compute the surviving resonant periodic orbits (big holes in Figure 1 left). Although everything we will see will be generic for any  $g(x, t)$ , we will fix this forcing from now on say to

$$g(x, t) = \sin(\omega t),$$

and  $T$  becomes  $T = \frac{2\pi}{\omega}$ .

It will be useful to write the system as a first order one by increasing the dimension. Let  $v = \dot{u}$ , then we can write it as

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\sin(u) + \varepsilon \sin(\omega t). \end{cases} \quad (2)$$

## 2 Dynamics of the unforced pendulum

### 2.1 Phase portrait

We will start with some exercises in order to understand the dynamics of the unperturbed pendulum ( $\varepsilon = 0$ ).

**Exercise 1.** Consider the unperturbed equations of the pendulum by setting  $\varepsilon = 0$  in Eq. (2):

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\sin(u) \end{cases} \quad (3)$$

1. Obtain the equilibrium points and state their type.
2. Obtain a Hamiltonian for system (3).

**Solution 1.** 1. The system possesses an elliptic equilibrium point at the origin and two saddle points at  $(\pm\pi, 0)$ .

2. The function

$$H(u, v) = \frac{v^2}{2} - \cos(v)$$

is the Hamiltonian of system (3) up to a constant value.

**Exercise 2.** Write a small program in Matlab to plot the phase portrait of system (3).

In this session we will focus on what happens with the periodic orbits inside the homoclinic loop when we add a small periodic forcing to system (3) (say  $\sin(\omega t)$ ). To this end, it will be crucial to compute the periods of the periodic orbits for  $\varepsilon = 0$ . Let's us now see a general theoretical approach to obtain a formula.

Assume we have Hamiltonian system of the form

$$H(u, v) = \frac{v^2}{2} + V(u),$$

where  $V(u)$  is a potential energy. Assume that we have a periodic orbit at level of energy  $\alpha \in \mathbb{R}$  given by  $H(u, v) = \alpha$ . Then, we can compute its period,  $T_\alpha$ , as follows:

$$\begin{aligned} T_\alpha &= \oint_{H(u,v)=\alpha} dt = \oint_{H(u,v)=\alpha} \frac{1}{\frac{du}{dt}} du \\ &= \oint_{H(u,v)=\alpha} \frac{1}{\dot{u}} du. \end{aligned}$$

using that the system is Hamiltonian and, hence,  $\dot{u} = \frac{\partial H(u,v)}{\partial v} = v$  we get

$$T_\alpha = \oint_{H(u,v)=\alpha} \frac{1}{v} du.$$

Isolating  $v$  from  $H(u, v) = \alpha$ ,

$$T_\alpha = \oint_{H(u,v)=\alpha} \frac{1}{\sqrt{2(\alpha - V(u))}} du.$$

But now we have two problems. First is that solving such an integral explicitly can be a nightmare, if possible at all. Second, doing integrals numerically is a difficult task, it's slow and imprecise. Fortunately, there is an alternative: compute a Poincaré map using a transversal section to the periodic orbit and capture the return time.

**Exercise 3.** Write a small program in Matlab to compute the periods of the periodic orbits using the Poincaré map from the section  $\{u = 0\}$  to itself.

## 2.2 Stroboscopic vs Poincaré map

In general, for autonomous systems, in order to study the existence of periodic orbits one typically considers the Poincaré map:

$$P : \Sigma \longrightarrow \Sigma,$$

where  $\Sigma$  is a co-dimension one section. In our case, we could take the vertical axis  $\Sigma = \{(u, v) \mid u = 0\}$ . Then, fixed points of the Poincaré map, points  $x_0 \in \Sigma$  such that  $P(x_0) = x_0$ , become initial conditions for periodic orbits for the flow, whose period is the flying time. Periodic points of the Poincaré map,  $P^n(x_0) = x_0$ , also give rise to periodic orbits for the flow, which cross  $n$  times the Poincaré section and their period becomes the addition of all flying times between consecutive impacts with the Poincaré section before  $x_0$  is reached again.

However, if the system is not autonomous (but periodic), then one needs to take

into account the initial time and consider Poincaré sections of the form  $\Sigma \times \mathbb{T}_T$ , with  $\mathbb{T}_T = \mathbb{R}/T\mathbb{Z}$ . Then, a sufficient condition for the existence of a periodic orbits becomes  $P^n(x_0, t_0) = (x_0, t_0 + mT)$ . That is, after the point  $x_0$  is reached again after  $n$  crossings and the total spent time is a multiple of  $T$ . Then, the flow possesses an  $mT$ -periodic orbit crossing the section  $n$  times. Note that, if total time spent to reach  $x_0$  again is not a multiple of  $T$ , then nothing can be said about the existence of a periodic orbit.

One of inconveniences of using Poincaré maps with non-autonomous systems is the need of computing the flying time. Although this can be done numerically, it requires extra computations than simply numerically integrating a flow, as one needs to compute the crossing with the section. Alternatively, one can use the stroboscopic map, which consists of integrating the system for a time  $T$ :

$$\begin{aligned} s: \Sigma_{t_0} &\longrightarrow \Sigma_{t_0} \\ x &\longmapsto \varphi_\varepsilon(T; x, t_0), \end{aligned}$$

where

$$\Sigma_{t_0} = \{(x, t) \in \mathbb{R}^2 \times \mathbb{T}_T\}, \mathbb{T}_T := \mathbb{R}/T\mathbb{Z}.$$

Then, if  $s^m(x_0) = x_0$ , then  $x_0$  is the initial condition for a  $mT$ -periodic orbit:

$$\varphi_\varepsilon(t_0 + mT; x_0, t_0) = x_0.$$

**Exercise 4.** *Write a small program in Matlab to compute the stroboscopic map. Write also a script to iterate it several times (say 100) for different initial conditions at the  $v$  axis. Note the resonances (probably you will need to play with  $\omega$  to observe things better). First use  $\varepsilon = 0$ , and then increase it a little bit and see what happens.*

### 3 The Melnikov method for subharmonic periodic orbits

Let us consider a planar field of the form

$$\dot{x} = f(x) + \varepsilon g(x, t), \tag{4}$$

where  $x = (u, v) \in \mathbb{R}^2$   $\varepsilon \geq 0$  is a small parameter and  $g(x, t)$  is  $T$ -periodic in  $t$ :

$$g(x, t + T) = g(x, t), \forall x \in \mathbb{R}^2.$$

For simplicity, let us assume that, for  $\varepsilon = 0$ , the unperturbed system

$$\dot{x} = f(x)$$

is Hamiltonian. That is, there exists a function  $H(u, v)$  such that

$$\begin{aligned} \dot{u} &= \frac{\partial H(u, v)}{\partial v} \\ \dot{v} &= -\frac{\partial H(u, v)}{\partial u}. \end{aligned}$$

Moreover, let us assume that the unperturbed system satisfies the following:

1. There exists a compact region completely covered by a continuum of periodic orbits.
2. Let  $T_\alpha$  be the period of the periodic orbit located at the energy level  $c$ :

$$\{(u, v) \in \mathbb{R}^2, H(u, v) = c\}.$$

Assume that  $\frac{d}{d\alpha}T_\alpha \neq 0$ .

Then we have the following result:

**Theorem 1.** *Assume the above conditions are satisfied. Assume that the unperturbed system has a periodic orbit,  $H(u, v) = \alpha$ , of period*

$$T_\alpha = \frac{m}{n}T, \quad (5)$$

where  $T$  is the period of the periodic forcing. Let  $x_0 = (u_0, v_0)$  be an initial condition for such a periodic orbit, and let

$$\varphi(t; x_0)$$

be the flow at such initial condition. Then let us define the so-called (subharmonic) Melnikov function

$$M(t_0) = \int_0^{mT} f(\varphi(t; x_0)) \wedge g(\varphi(t; x_0), t + t_0) dt. \quad (6)$$

Then, if there exists  $\bar{t}_0$  such that

1.  $M(\bar{t}_0) = 0$
2.  $M'(\bar{t}_0) \neq 0$ ,

then, for  $\varepsilon > 0$  small enough, the perturbed system possesses a periodic orbit of period  $mT$  with initial condition  $x_\varepsilon = x_0 + O(\varepsilon)$  at  $t = \bar{t}_0$ :

$$\varphi_\varepsilon(\bar{t}_0 + mT; \bar{t}_0, x_\varepsilon) = x_\varepsilon.$$

**Exercise 5.** *Compute the analytical expression of the Melnikov function for system (3) (you don't have to do the integral!).*

**Exercise 6.** *Write a little program in Matlab to numerically compute the Melnikov integral obtained in exercise 5.*

## 4 Numerical computation of hyperbolic periodic orbits: stroboscopic map

We now want to numerically compute the initial conditions  $(x_\varepsilon)$  given by Theorem 1. Here we describe a Newton method to do that in a general setting:  $n$ -dimensional not necessary Hamiltonian systems.

Assume we want to compute a periodic orbit of an  $n$ -dimensional non-autonomous periodic system of the form

$$\dot{x} = f(x, t), \quad (7)$$

where  $f(x, t)$  is a  $T$ -periodic field in  $t$ :  $f(x, t + T) = f(x, t)$ , for any  $x \in \mathbb{R}^n$ . Due to the periodicity, instead of using Poincaré maps in the state space it will be much more convenient to use the so-called stroboscopic map, which is indeed a Poincaré map but using a section in time. This map is given by flowing system (7) for a time  $T$  with initial condition  $(x_0, t_0)$ :

$$s(x_0) = \varphi(t_0 + T; x_0, t_0).$$

Provided that system (7) is  $T$ -periodic in  $t$ ,  $s(x)$  becomes a map from the time section  $t = t_0$  to itself:

$$\begin{aligned} s : \Sigma_{t_0} &\longrightarrow \Sigma_{t_0} \\ x &\longmapsto \varphi(t_0 + T; x, t_0), \end{aligned}$$

where

$$\Sigma_{t_0} = \{(x, t) \in \mathbb{R}^2 \times \mathbb{T}_T\}, \quad \mathbb{T}_T := \mathbb{R}/T\mathbb{Z}.$$

Imagine that we want to compute the initial condition,  $x^p$  for a periodic orbit of system (7) at the section  $\Sigma_{t_0}$ . Recalling its periodicity, the period of such a periodic orbit must be a multiple of  $T$ , say  $mT$ . In other words, we must look for periodic points of the map  $s$ , that is, points  $x^p$  such that

$$s^m(x^p) = x^p.$$

One of the most extended methods for numerically computing such points is the Newton method, assuming that one has some idea about where such a points lies, as we need a first approximation for the Newton method to converge where we want.

#### 4.1 Newton method for hyperbolic fixed points

Let us assume that we are looking for a  $T$ -periodic orbit; that is,  $m = 1$  and we want to find a fixed point of the map  $s$ . Then, we want to solve the equation

$$F(x) := s(x) - x = 0.$$

The Newton method consists of considering the linear approximation of the function  $F$  around some point  $(x_0, F(x_0))$  (which is a first approximation of the solution we are looking for) and solve the linear system instead. This provides a new point,  $(x_1, F(x_1))$  which, hopefully, is a more accurate solution than the initial one.

The linear approximation of equation  $F(x) = 0$  around  $(x_0, F(x_0))$  becomes

$$F(x_0) + DF(x_0)(x - x_0) = 0, \quad (8)$$

where

$$DF(x_0) = D_x s(x_0) - I.$$

In Section 4.3 we will see how to compute the differential  $D_x s(x)$ .  
If we solve Equation (8) for  $x$  we get

$$x = x_0 - (DF(x_0))^{-1} F(x_0).$$

This leads to an iterative process

$$x_{i+1} = x_i - (DF(x_i))^{-1} F(x_i), \quad (9)$$

which converges quadratically to  $x^p$  provided that  $x_0$  is a good enough approximation.

Alternatively, some programming environments (like Matlab) offer routines to solve linear equations which might be more efficient than computing the inverse  $(DF(x_0))^{-1}$ . In this case, the linear system to be solved would be

$$DF(x_i)x_{i+1} = DF(x_i)x_i - F(x_i). \quad (10)$$

**Remark 1.** *Note that the Newton method requires the matrix  $DF(x_i)$  to be invertible. This implies two things:*

- *$DF$  must be invertible at the starting point*
- *$DF$  must be invertible at the fixed point! This implies that the Newton method will have troubles if the fixed point we are looking for is a center, as the eigenvalues of  $Ds$  would have real part equal to one. In other words, the periodic orbit has to be hyperbolic.*

## 4.2 Newton method for (hyperbolic) periodic orbits

Similarly we can apply the Newton method to solve the equation

$$F(x) = s^m(x) - x$$

to get the same expression as in Eq. (9). However, in this case the computation of  $DF$  becomes now a bit more tricky. Using that  $s^i(x) = s(s^{i-1}(x))$ , we apply the chain rule to get

$$DF(x_i) = \prod_{j=i}^{j=1} Ds(s^{j-1}(s)),$$

so we need to evaluate the differential  $Ds(x)$  at the points  $s^j(x)$  for  $j = 1..i$ .

Although there is nothing wrong with this approach from the theoretical point of view, in next section we will see a numerical method to compute  $Ds(x)$  which makes the computation of  $Ds^m(x)$  straightforward, with needing to multiply matrices (see Remark 4 below).



### 4.3 Computation of the differential of the stroboscopic map: the variational equations

Now the question arises, how do we compute  $Ds$ ? Note that the flow  $\varphi(t; x_0, t_0)$  is straightforward to differentiate with respect to  $t$ , as one recovers the field  $f$ , but we need to differentiate it with respect to the initial condition  $x_0$ ! But we can do the following. Applying the fundamental theorem of calculus and the definition of the flow, we can write

$$\begin{aligned} s(x) &= \varphi(t_0 + T; x, t_0) = x + \int_{t_0}^{t_0+T} \frac{d}{dt} \varphi(t; x, t_0) dt \\ &= x + \int_{t_0}^{t_0+T} f(\varphi(t; x, t_0), t) dt. \end{aligned}$$

If we now differentiate with respect to  $x$ , we get

$$Ds(x) = I_n + \int_{t_0}^{t_0+T} D_x f(\varphi(t; x, t_0), t) D_x \varphi(t; x, t_0) dt, \quad (11)$$

where  $I_n$  is the  $n \times n$  identity matrix and we write  $D_x$  to emphasize that we differentiate with respect to  $x$ . Again, by applying the fundamental theorem of calculus backwards, we realize that Equation (11) is the solution of the differential equation

$$\frac{d}{dt} D_x \varphi(t; x, t_0) = D_x f(\varphi(t; x, t_0)) D_x \varphi(t; x, t_0), \quad (12)$$

at  $t_0 + T$ . Equation (12) is called the (first) variational equation.

Some remarks:

**Remark 2.** If  $f(x, t)$  is a field in  $\mathbb{R}^n$ , then this equation becomes an  $n \times n$ -dimensional differential equation.

**Remark 3.** Equation (12) is evaluated along the flow  $\varphi(t; x, t_0)$ , which is unknown. Hence, this equation needs to be solved together with the equation  $\dot{x} = f(x, t)$ , leading to a system of dimension  $n + n \times n$  with initial condition  $(x, I_n)$ .

**Remark 4.** If we want to compute  $Ds^m(x)$ , we just need to integrate the variational equations from  $t_0$  to  $t_0 + mT$ !

**Exercise 7.** Write a program in Matlab in order to compute a fixed point of the stroboscopic map. Note that the initial seed will be taken from a zero of the Melnikov function, which may have several. You will need to tune  $\omega$  to guarantee that this fixed point exists. Compute the eigenvalues of  $Ds$  at the fixed points to tell their type.

## 5 Numerical computation of non-hyperbolic periodic orbits: Poincaré map

As noted in Remark 1, the Newton method to find fixed points (or periodic orbits) of the stroboscopic map will fail if that one is non-hyperbolic: that is, the associated eigenvalues of the stroboscopic map have real part equal to 1. Alternatively, we can use a Poincaré map using a section in the state space. This method is more robust in that sense, but, as we will show below, the computation of the differential becomes slightly more tricky.

Let us consider the Poincaré map

$$P_\varepsilon(v, t) : \begin{array}{ccc} \{u = 0\} \times \mathbb{R} & \longrightarrow & \{u = 0\} \times \mathbb{R} \\ (0, v_0, t_0) & \longmapsto & (0, \varphi_1(t^*; 0, v_0, t_0), t^*), \end{array} \quad (13)$$

where

$$\varphi_\varepsilon(t; x_0, t_0) = \begin{pmatrix} \varphi_1(t; x_0, t_0) \\ \varphi_2(t; x_0, t_0) \end{pmatrix} \quad (14)$$

is the solution of the perturbed system (1) with initial condition  $x_0$  at  $t = t_0$  and  $t^*$  is such that<sup>1</sup>

$$\varphi_1(t^*; 0, v_0, t_0) = 0 \quad (15)$$

for the second time, as we consider a return map in the direction that the section is left.

Recall that system (1) is non-autonomous. Hence, the initial time  $t_0$  matters and we have to carry it on. From now on, we will abuse notation in Equation (13) and omit writing the first coordinate  $u$ , as it takes the 0.

Taking into account the periodicity of the non-autonomous system (1), an initial condition at  $(0, v_0) \in \{u = 0\}$  for  $t = t_0$  of a periodic orbit of the full system will need to satisfy

$$P^n(v_0, t_0) = \begin{pmatrix} v_0 \\ t_0 + mT \end{pmatrix} \quad (16)$$

for some  $n, m > 0$ . Indeed, the integers  $n$  and  $m$  are the same as in previous sections, made explicit through the congruency equation (5). Here one clearly see that the role of  $n$  is to count the “loops” that a periodic orbit of period  $mT$  makes around the origin.

Applying the Newton method to the equation

$$F(v_0, t_0) = P^n(v_0, t_0) - \begin{pmatrix} v_0 \\ t_0 + mT \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (17)$$

and arguing as in Section 4 we get the iterative process

$$(v_{i+1}, t_{i+1}) = (v_i, t_i) - (DF(v_i, t_i))^{-1} F(v_i, t_i), \quad (18)$$

---

<sup>1</sup>Here we assume that we know how to compute  $t^*$ . It can be easily done using a Newton method to solve Equation (15). Below we show how to compute the necessary derivatives for the Newton method. Otherwise, it can be computed using the “Events” functionality of Matlab.

where now

$$DF(v_i, t_i) = DP^n(v_i, t_i) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We now wonder, how do we compute  $DP^n$ ? Let us see it for  $n = 1$  first.

Recalling that  $\varphi_\varepsilon = (\varphi_1, \varphi_2)$ , we get

$$DP(v_0, t_0) = \begin{pmatrix} D_{v_0}\varphi_2(t^*; 0, v_0, t_0) & D_{t_0}\varphi_2(t^*; 0, v_0, t_0) \\ \frac{\partial}{\partial v_0}t^* & \frac{\partial}{\partial t_0}t^* \end{pmatrix}.$$

Note that, in the first row, we have written the total derivatives  $D_{v_0}$  and  $D_{t_0}$  instead of partial ones,  $\partial/\partial v_0$  and  $\partial/\partial t_0$ , because  $t^*$  actually depends on  $v_0$  and  $t_0$  through Equation (15). Let us compute such total derivatives.

For the first one we get

$$D_{v_0}\varphi_2(t^*; 0, v_0, t_0) = \frac{\partial}{\partial v_0}\varphi_2(t^*; 0, v_0, t_0) + \varphi_2'(t^*; 0, v_0, t_0)\frac{\partial t^*}{\partial v_0}.$$

Let us now see how to compute all the terms appearing in this equation.

On one hand,

$$\varphi_2'(t^*; 0, v_0, t_0) = f_2(\varphi_\varepsilon(t^*; 0, v_0, t_0)) + \varepsilon g_2(\varphi_\varepsilon(t^*; 0, v_0, t_0), t^*),$$

where  $f_2() + \varepsilon g_2()$  is the second coordinate of the field evaluated at the image of the Poincaré map.

On the other one,  $\frac{\partial \varphi_2(t^*; 0, v_0, t_0)}{\partial v_0}$  is given by integrating the variational equations from  $t_0$  to  $t^*$ , as we are differentiating with respect an initial condition.

What about  $\frac{\partial t^*}{\partial v_0}$ ? Recall that, for given  $v_0$  and  $t_0$ ,  $t^*$  (which we somehow know how to compute) solves Equation (15). Therefore, assuming that

$$\varphi_1'(t^*; 0, v_0, t_0) \neq 0, \tag{19}$$

we can use the Implicit Function Theorem to get

$$\begin{aligned} \frac{\partial t^*}{\partial t_0} &= -\frac{\frac{\partial}{\partial t_0}\varphi_1(t^*; 0, v_0, t_0)}{\varphi_1'(t^*; 0, v_0, t_0)} \\ &= -\frac{\frac{\partial}{\partial t_0}\varphi_1(t^*; 0, v_0, t_0)}{f_1(\varphi_\varepsilon(t^*; 0, v_0, t_0)) + \varepsilon g_1(\varphi_\varepsilon(t^*; 0, v_0, t_0), t^*)}. \end{aligned} \tag{20}$$

Note that, condition (19) is satisfied, as the flow is transversal to the section  $\{u = 0\}$ . If it were tangent, then we would have a problem, of course!

The denominator of the last equation is just the first coordinate of the perturbed field evaluated at the image of the Poincaré map.

Regarding  $\frac{\partial}{\partial t_0}\varphi_1(t^*; 0, v_0, t_0)$ , note that it is a derivative with respect to the initial time,  $t_0$ . Adding time as a variable, this can be computed integrating the variational equations of the system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -\sin(u) + \varepsilon \sin(\omega s) \\ \dot{s} &= 1, \end{aligned} \tag{21}$$

where now  $s$  plays the role of time and  $t_0$  becomes  $s_0$ .

Proceeding similarly, the elements of the second row of  $DP$  become

$$\frac{\partial t^*}{v_0} = - \frac{\frac{\partial}{\partial v_0} \varphi_1(t^*; 0, v_0, t_0)}{f_1(\varphi_\varepsilon(t^*; 0, v_0, t_0)) + \varepsilon g_1(\varphi_\varepsilon(t^*; 0, v_0, t_0))},$$

and  $\frac{\partial t^*}{\partial t_0}$  is already given in Equation (20)

The differential  $DP^n(v_0, t_0)$  can be computed proceeding similarly as in Section 4.2, multiplying  $DP$  evaluated at the iterates  $P^n(v_0, t_0)$  or integrating the previous variational equations until the  $2n$ -th impact with the section  $\{u = 0\}$  occurs.

**Exercise 8.** *Compute analytically the variational equations of system (21).*

**Exercise 9.** *Write a program in Matlab to compute initial conditions for periodic orbits by performing a Newton method to Equation (17). For that you will need to slightly modify the programs you wrote in the previous exercises.*

## References

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